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Determination of the Nucleon-Nucleon Elastic Scattering Matrix. I. Phase-Shift Analysis of Experiments Near 140 MeV*

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A phase-shift analysis has been carried out for (p,p) and (n,p) experiments near 140 MeV. The two kinds of data were analyzed separately and combined (with charge independence assumed). Investigations were made on the accuracy of the data, the search procedure, the significance of error-matrix calculations, the significance of the absolute value of x^2 (the least-squares sum), and the importance of using an energy dependence for the phase shifts and for the data. A single solution was obtained that gives a good fit to both the (p,p) and the (n,p) data. The gross features of charge independence were found to be accurately verified. Determinations of the pion-nucleon coupling constant g^2 and the pion mass gave results consistent with previous nucleon-nucleon analyses. The (n,p) data, although they include five kinds of experiments, are not complete enough by themselves to permit an accurate phase-shift analysis. The (p,p) data are complete. The combined (p,p) plus (n,p) data give $T=1$ phase shifts that agree with the phase shifts from the (p,p) analysis alone, and they give accurate values for the $T=0$ phase shifts.

I. INTRODUCTION

A PHASE-SHIFT analysis is an attempt to translate experimental measurements (observables) into well-determined scattering amplitudes, since these are the quantities that can be readily compared with theoretical predictions. In this sense, the phase-shift analysis should contain as little theory as possible. The scattering amplitudes (or phase shifts) constitute an experimental statement, and the phase-shift analysis should logically be done by the experimental groups who measure the observables.

Historically, the phase-shift analysis of nucleon-nucleon scattering was first done at single energies.[1] The low-angular-momentum (low-l) phase shifts were adjusted to fit the observables, with the high-l phase shifts being set equal to zero. Later it was found that an improvement is obtained if the high-l phase shifts are calculated from theory (the one-pion-exchange contribution, OPEC) instead of being set equal to zero.[2] This "modified phase-shift analysis" is thus a mixture of experiment and theory, but the theoretical contribution to the amplitudes is rather small and is well-determined.

The next development in phase shift analyses was to use an energy-dependent analysis and fit nucleon-nucleon data at all energies in the elastic scattering region (0 to about 400 MeV) simultaneously. In the Yale energy-dependent analyses,[3,4] essentially arbitrary forms (which used potential-model starting points) represented the energy-dependent phase shifts. The idea here was to keep the amount of theory in the phase-shift analyses to a minimum so that the resulting phase shifts would be a representation of the experimental data and would not depend on any specific theoretical model. In the Livermore energy-dependent analysis,[5] the phase shifts were assigned forms similar to dispersion-theoretic forms. Thus more theory was put into the phase-shift analysis, and the resultant phase shifts reflected in part the kind of theoretical form used for each particular phase-shift determination. In this manner it was hoped to obtain information about the phase-shift parameters that would relate (for example) to discontinuities in a Mandelstam diagram. The Livermore analysis[5] was for proton-proton scattering only, while the Yale analyses[3,4] included the neutron-proton system as well.

The energy-dependent analyses were successful in establishing the existence of a single phase-shift solution type (solution 1 of Stapp[1]) that gave a good fit to nucleon-nucleon data in the entire elastic region. (These analyses did not actually yield a single solution but rather a family of solutions, all of the same general type.) The energy-dependent analyses were useful in relating various experiments and in establishing the consistency (or inconsistency) of the measurements. From the standpoint of providing direct theoretical information, the Livermore analysis was not completely

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[2] M. J. Moravcsik, University of California Radiation Laboratory Report UCRL 5317-T, 1958 (unpublished); A. F. Grashin, Zh. Eksperim. i Teor. Fiz. **36**, 1717 (1959) [English transl.: Soviet Phys.—JETP **9**, 1223 (1959)]; P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959); **116**, 1248 (1959); M. H. MacGregor and M. J. Moravcsik, Phys. Rev. Letters **4**, 524 (1960); M. H. MacGregor, M. J. Moravcsik, and H. P. Noyes, Phys. Rev. **123**, 1835 (1961).

[3] G. Breit, M. H. Hull, Jr., K. E. Lassila, K. P. Pyatt, and H. M. Ruppel, Phys. Rev. **128**, 826 (1962). This work was first reported at the London "Few Nucleon Conference" in 1959.

[4] M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **128**, 830 (1962).

[5] H. P. Stapp, H. P. Noyes, and M. J. Moravcsik, in *Proceedings of the 1962 Annual International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 131, and in *Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester*, edited by E. C. G. Sudershan, J. H. Tinlot and A. C. Melissions (Interscience Publishers, Inc., New York, 1960), p. 128.

successful. The parameters of the functional forms that were used to represent the phase shifts do not relate in any direct manner to Mandelstam discontinuities.

From the point of view of a precise determination of nucleon-nucleon scattering amplitudes, with as little theory added in as possible, one should work at a series of energies, doing each energy independently from the rest. Many single-energy analyses have already been carried out for the proton-proton system[1,2,6,7] and more recently for the proton-proton plus proton-neutron system.[8–10] In the present analysis we have selected the energy region near 140 MeV. This region was somewhat neglected in some of the earlier analyses[2] because there were experimental disagreements in the proton-proton data (the famous Harvard-Harwell D discrepancy). The Harvard and Harwell (p,p) experiments have now been brought into reasonable agreement (except for cross-section differences), and in addition new measurements on the neutron-proton system have become available. Early phase-shift analyses in this energy range were performed at Dubna[10] and at Harwell.[9] Both of these latter analyses were in a sense preliminary. The Dubna analysis did not include the complete set of (p,p) and (n,p) data. Also, the $l=4$ waves (and higher) were taken from OPEC, an approximation that is not quite accurate enough for the (p,p) system (where the data are most precise and complete). And the Dubna analysis used the Harvard differential cross section,[11] which has difficulties associated with it. The recent Harwell analysis was based on the Harwell (p,p) cross-section data,[12] about which there is some question as to the angular distribution. (The nucleon-nucleon data are discussed in detail in Sec. II.) Also the $T=1$ phase shifts in the Harwell analysis were not varied to adjust to the (n,p) data, but were held fixed at the values obtained from fitting (p,p) data alone.

We have taken the seven kinds of measurements available for the (p,p) system $(\sigma_t,\sigma(\theta),P,D,R,A,R')$, and the six kinds of measurements available for the (n,p) system $(\sigma_t,\sigma(\theta),P,D,R,A)$, and have treated the (p,p) data separately, the (n,p) data separately, and the combined (p,p) plus (n,p) data together (assuming charge independence). We have investigated the following:

1. Uniqueness of the solution.
2. Effect of different data selections.
3. Significance of the absolute value of the least-squares sum χ^2.
4. Energy-dependent effects in the data selection and in the phase shifts.
5. The necessity for over-all normalization constants for some of the data.
6. Determination of the pion-nucleon coupling constant g^2 and the pion mass μ.
7. Charge independence—a comparison of $T=1$ phase shift and g^2 values from different data selections.
8. Error-matrix calculations for the phase shifts and coupling constant.

Finally we compare these results with the previous single-energy phase-shift determinations and with the energy-dependent results at 140 MeV.

II. DATA SELECTION

The data used in this analysis[11–34] are listed in Table I. The following comments about the data should be noted.

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TABLE I. Nucleon-nucleon scattering data near 140 MeV.

Observable	Laboratory	Energy (MeV)	Reference
	(p,p) Data		
σ_{total}	Harvard	147.2	13
$\sigma(\theta)$	Orsay	155	14
$\sigma(\theta)$	Harvard	147	11
$\sigma(\theta)$	Harwell	142	12
$P(\theta)$	Harvard	147	11
$P(\theta)$	Harwell	142	12
$D(\theta)$	Harvard	142	15
$D(\theta)$	Harwell	143	16
$R(\theta)$	Harvard	140	17
$R(\theta)$	Harwell	142	18
$A(\theta)$	Harvard	139	19
$A(\theta)$	Harwell	143	20
$R'(\theta)$	Harvard	137.5	21
$R'_t(\theta)$	Harwell	140	22
	(n,p) Data		
σ_{total}	Harwell	126, 153	23
σ_{total}	Rochester	140, 156	24
$\sigma(\theta)$	Harvard	128	25
$\sigma(\theta)$	Harwell	130	26
$\sigma(\theta)$	Harwell	137	27
$\sigma(\theta)$	Harwell	153	28
$P(\theta)$	Harvard	128	25
$P(\theta)$	Harvard	126	29, 35
$P(\theta)$	Harwell	140	30
$P(\theta)$	Harvard	143	31, 34
$D_t(\theta)$	Harvard	128	32
$R(\theta)$	Harvard	140	33, 34
$A(\theta)$	Harvard	135	33, 34

(p,p) Differential Cross-Section Data

The (p,p) total cross-section measurements[13] showed that the normalization of the Harvard differential cross-section measurements[11] was in error by about 8%. Also, the phase-shift analysis of Palmieri and Prenowitz[7] and the Livermore energy-dependent analysis[5] showed that the small-angle data (runs H14 of Ref. 11) may possibly be about 6% lower than the large-angle data (runs H1 and H8). Hence the Harvard differential cross-section data[11] were handled in the present analysis by multiplying run H14 by 0.87 and multiplying runs H1 and H8 by 0.93. These runs were then each given a separate normalization parameter that could vary during the phase-shift search, but the three normalization parameters were tied together in the sense that the integral over the differential cross section was subtracted from the measured total cross section,[13] and the difference, weighted by the experimental error (1%) on the total cross section, was entered as a term in the χ^2 sum. Although we used a lower normalization for the small-angle Harvard differential cross-section data than for the large-angle data, there is no way on the basis of the present analysis to determine if this is actually correct. We could have left the normalizations the same, and the phase shifts would have changed slightly to accommodate the altered data, with χ^2 being essentially unchanged. The Harwell differential cross section,[12] when integrated, agrees with the Harvard total cross section,[13] so that it is normalized correctly. However, the Harwell angular distribution does not agree with the Harvard angular distribution. The recent Orsay differential cross section[14] has the correct normalization. Figures 1–3 show the Harvard (renormalized as outlined above), Harwell and Orsay differential cross sections. Inasmuch as the shapes of the Orsay and (renormalized) Harvard differential cross sections agree, we conclude that the Harwell angular distribution is probably incorrect. In the final (p,p) analysis, both the Harvard and Harwell cross sections were discarded, and only the Orsay measurement was used. This was a major factor in our decision to introduce an energy dependence into the phase-shift analysis, since the Orsay measurement was at a higher energy, 155 MeV, than any of the Harvard or Harwell (p,p) measurements.

The differential cross-section data can also be discussed in terms of the least-squares fits to the data that were obtained. The Harvard data consisted of 37 points (the 4° c.m. point was omitted) that were in three separate runs with separate normalization constants. The contribution to the least-squares sum χ^2 was 16 for a typical phase-shift solution. Hence the data are internally consistent. The Harwell data consisted of four runs, each of which was separately normalized, totaling 31 points (the 5.2° point was arbitrarily omitted). The normalizations were constrained to match the total cross section (except for the smallest angle points), as described for the Harvard data above. The best fit to the Harwell differential cross-section data gave a χ^2 contribution of 95 for the 31 points (the 6.2° point contributed 18 to this sum). We would expect a χ^2 contribution of around 30. Hence the Harwell data do not seem to be consistent among themselves to within the quoted relative errors (see Fig. 2). The Orsay data consisted of 23 points (the point at 8.3° was arbitrarily omitted) normalized in one run. For the best fit, χ^2 was 32. However, the two points at 22.9 and 25° con-

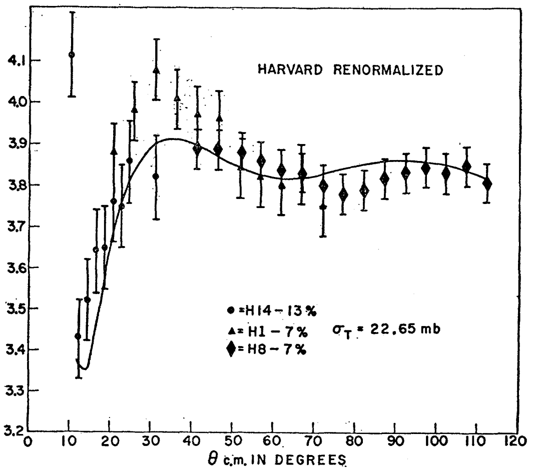

FIG. 1. Harvard differential cross-section data at 147 MeV. The data are from Ref. 11 and have been renormalized downward by the amounts shown in the figure (see Sec. II). The solid curve is a phase-shift solution fit to the data.

tributed about half of this total. Omitting them gives a χ^2 contribution of 15. (Since there is nothing to indicate that those points are "wrong," it is not clear that omitting them improves the physical content of the data.)

The point we are trying to make in this discussion of the least-squares fitting of differential cross-section data is that the least-squares sum χ^2 that is obtained depends quite radically on how much data averaging was done by the experimenters and by the person doing the phase-shift analysis. The Harvard, Harwell, and Orsay cross sections all give almost the same phase shifts (as we will show in Sec. IV), and yet the χ^2 values differ drastically. By throwing away data points, we can adjust χ^2 until it has about the expected value, but it is not clear that in doing this we are improving the physics (the phase-shift values). We feel that the absolute values of χ^2 is not very meaningful when one considers the present status of nucleon-nucleon scattering data, and to assign (say) a 50% confidence value to a certain χ^2 sum is to read an accuracy into the experimental errors that simply is not there in most cases.

(*p*,*p*) Polarization Data

Polarization data should be assigned both relative and over-all normalization errors,[9] as was done for the differential cross sections above. The over-all error here is the uncertainty in the initial beam polarization. For the Harvard data,[11] 28 data points were used as averaged by Wilson.[35] An over-all normalization error of 3% was assigned, and the 4° c.m. point was omitted. The average χ^2 value obtained was about 1 per data point, and the over-all normalization arrived at in the analysis was 0.98. For the Harwell polarization data,[12] the 5.2° point was arbitrarily omitted, and 29 data points were used, with an over-all normalization error of 2.2% being assigned. The points at 6.2 and 78.05° each contributed 13 to χ^2 and hence should be omitted if a reasonable looking χ^2 value is desired. Again it is a question as to whether omitting these points constitutes any improvement in the phase-shift values. With these points omitted, the average χ^2 value per data point was close to one, and the over-all normalization value obtained in the present phase-shift analysis was about 0.97–0.98.

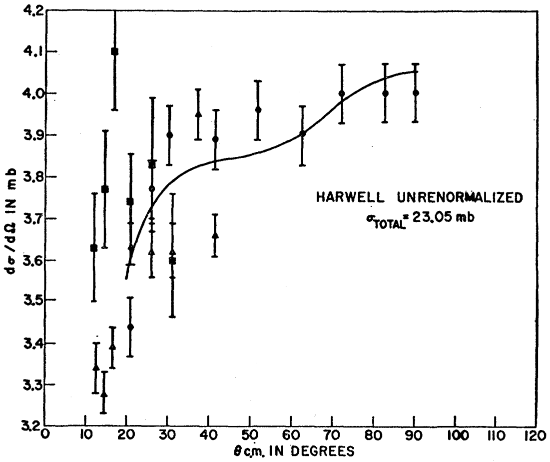

FIG. 2. Harwell differential cross-section data at 142 MeV. The data are from Ref. 12 (see Sec. II). The solid curve is a phase-shift solution fit to the data.

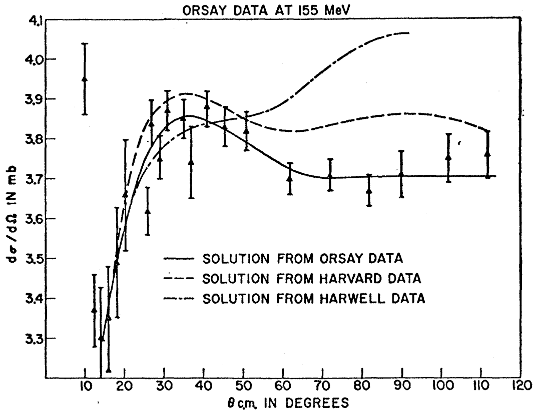

FIG. 3. Orsay differential-cross-section data at 155 MeV. The data are from Ref. 14. The solid curve is a phase-shift solution fit to the data, and the dashed curves are taken from Figs. 1 and 2 to illustrate how the shapes of the different measurements compare (see Sec. II for a discussion).

(*p*,*p*) Triple Scattering Data

The *D*, *R*, *A*, and *R'* measurements at Harvard and Harwell are all in reasonable agreement. In the phase-shift analysis, the Harvard *A* and *R'* data were assigned over-all normalization errors of 4% and 5%, respectively. The preliminary Harwell *R'* value[22] at 72° seems to be inconsistent with the other Harwell and Harvard *R'* data points, and it was omitted.

(*n*,*p*) Differential Cross-Section Data

Four sets of data were combined.[25–28] Each set had its own normalization constant and was constrained to agree with total cross-section measurements[23,24] to within experimental errors.

(*n*,*p*) Polarization Data

The Harvard[31,34] and Harwell[30] polarization measurements near 140 MeV were assigned over-all normalization errors of 3.5% and 4.4%, respectively. The Harvard data[31] were used as corrected for deuteron binding effects by Cromer and Thorndike,[34] and the correction error was combined quadratically with the

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[35] R. Wilson, *The Nucleon-Nucleon Interaction* (Interscience Publishers, Inc., New York, 1963).

TABLE II. Data selections used in the present phase-shift analysis.

Data designation	(p,p) Data selections
Harvard	Harvard σ, P, D, R, A, R' data were used at 147 MeV. Data taken at energies other than 147 MeV were adjusted slightly to allow for the energy difference.
Harwell	Harwell σ, P, D, R, A and Harvard R' were used at 142 MeV. No energy-shifting changes were made in the data.
Set A	Orsay σ, Harvard and Harwell P, D, R, A, and Harvard R' were included. Each set of data was used at its experimental energy, and an energy-dependent search was carried out over the narrow band of energies from 137.5 to 155 MeV.
Set B	*Set A* plus the Harvard σ (renormalized).
Set C	*Set A* plus the Harvard σ (renormalized) and the Harwell σ.
Set A'	*Set A* with a few data points removed that were more than two standard deviations away from the least-squares average, and with the Harwell R' data added. These data are listed in Table XI.
142 MeV (p,p)	*Set A* with all of the data shifted in energy (using the results of the energy-dependent analysis) to be at 142 MeV. This set was used for a monoenergetic error-matrix calculation.
	(n,p) Data selections
137 MeV (n,p)	All of the (n,p) data listed in Table I were used except for Harvard 128-MeV P measurements. (Refs. 25, 29.) The energy was called 137 MeV, and no data energy-shifting changes were made. The four differential cross section (Refs. 25–28) were normalized (separately) to match the 137-MeV total cross-section value.
Set E	All of the (n,p) data listed in Table I were used at the experimental energies, and an energy-dependent analysis was carried out over the range from 128 to 153 MeV. The differential cross sections (Refs. 25–28) were each normalized to a total cross-section value at the appropriate experimental energy.
Set E'	*Set E* with $P(\theta)$ data at 128 MeV (Ref. 25) removed. These data are listed in Table XI.
142 MeV (n,p)	This is the counterpart of the *142-MeV* (p,p) data selection and is based on *Set E*.
	Combined (p,p) plus (n,p) data selections
Set D	*Set A* plus *Set E*.
Set D'	*Set A'* plus *Set E'*. See Table XI.
142 MeV (p,p) $+(n,p)$	*142 MeV* (p,p) plus *142 MeV* (n,p).

statistical errors. The Harvard polarization measurements at 128 MeV[25,29] carried normalization errors of 8 and 4%, respectively. The former set of data (Ref. 25) required a normalization of 10%, and the contribution to χ^2 was about 3.5 per data point. Hence in the final calculations these data were omitted.

(n,p) Triple Scattering Data

The depolarization transfer data[32] were used directly, while the R and A data[33] were used as corrected for binding by Cromer and Thorndike.[34]

A number of different data selections were used in the present work. These are labeled and described in Table II. The phase shifts used in this analysis are Stapp nuclear bar phase shifts.[1] The phase shifts not treated as free parameters were calculated from OPEC up to $l=18$. OPEC was used in this open form rather than as a closed expression so that Coulomb phase shifts could be correctly included for the (p,p) system.

III. UNIQUENESS OF THE PHASE-SHIFT SOLUTION

The (p,p) data are complete enough that a unique determination of the scattering matrix should be obtainable. The (n,p) data are not complete, however, since the triple scattering parameters D_t, R, and A, which are needed over the whole angular range 0–180°, have been measured only over narrow angular regions. Hence an analysis of the (n,p) data alone is not expected to yield a unique solution, but the (n,p) data combined with the (p,p) data might be expected to give a unique result if charge independence is assumed.

A number of investigations have already established the uniqueness of the (p,p) solution at 147 MeV.[7,9,10] As a quantitative check on this conclusion, we used the *Harvard* data set and started twenty problems from random phases, using 9 phases ($l_{\max}=3$) in the search. Solution 1 of Stapp,[1] with the least-squares sum $\chi^2=94$, was found once; solution 2 of Stapp, with $\chi^2=162$, was found four times; and a third solution, with $\chi^2=249$, was found once. All other solutions had $\chi^2>400$, and they could not be continued into any of these three solutions by using a variety of search procedures. As we have stated above, we do not feel that the absolute value of χ^2 is a particularly meaningful concept, since it depends so much on how the data selection is made. However, once the data selection is fixed, the relative value of χ^2 from one solution to another has statistical significance. In the present case solution 2 represents a much poorer fit to the data than does solution 1. In detail, both solution 1 and solution 2 fit the differential cross section and polarization data equally well ($\chi^2=59$ for both cases); but for the sum of D, R, A, and R', solution 1 gives $\chi^2=35$ and solution 2 gives $\chi^2=103$. Also solution 2 forces a much larger renormalization on the data (especially the parameter A) than does solution 1. Hence solution 2 should be ruled out. Other investigations[3,5,36] have also ruled out solution 2.

Solutions 1, 2, and 3 were also calculated using the *Harwell* data set with the nine-parameter phase-shift search, giving χ^2 values of 289, 342, and 394, respectively. When calculated from the *Harvard* data set with *11 phase shifts* in the search (see discussion below), solutions 1–3 gave χ^2 values of 62, 156, and 248. We conclude again that the (p,p) data near 147 MeV are of sufficient accuracy and completeness to rule out all solutions except solution 1.

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[36] M. J. Moravcsik and Riazuddin, Phys. Letters 4, 243 (1963).

TABLE III. (p,p) phase shifts for different data selections.

Data selection	Harvard	Harwell	*Set A*	*Set B*	*Set C*
No. of data points	91	93	126	163	193
χ^2	62	186	163	194	347
g^2	14.4	14.4	14.4	14.4	14.4
Energy (MeV)	147	142	137.5–155	137.5–155	137.5–155
$^1S_0$	15.12	15.24	15.66	15.33	15.49
$^1D_2$	5.24	4.62	4.82	4.98	4.96
$^1G_4$	0.70	0.78	0.73	0.70	0.71
$^3P_0$	5.13	6.08	5.72	5.65	5.80
$^3P_1$	−17.75	−17.22	−17.61	−17.64	−17.61
$^3P_2$	13.86	14.13	14.09	14.00	13.97
ϵ_2	−2.61	−2.51	−2.61	−2.62	−2.64
$^3F_2$	0.44	0.17	0.57	0.50	0.27
$^3F_3$	−1.86	−1.82	−1.87	−1.89	−1.93
$^3F_4$	1.02	0.55	1.02	1.02	0.83
ϵ_4	−0.59	−0.45	−0.62	−0.58	−0.56

The (n,p) data are much less complete than the (p,p) data near 140 MeV. Data set *137 MeV* (n,p) was used to investigate the uniqueness of (n,p) phase-shift solutions. A series of 15 random starts, using first 16 phase shifts, and later only 12 phases in the search, gave 15 different answers in each case, with χ^2 values ranging from 100 to 300. It was also found that a solution 1 type start would tend to stay near solution 1. We can picture the (n,p) χ^2 versus phase-shift surface as being very flat and gently undulating, so that the random starts each end up in a different local minimum. The minimum corresponding to solution 1 seems to be no lower than other nearby local minima. Hence the (n,p) data alone can give some information about solution 1 only if its location is accurately known in advance, but they do not provide any direct information as to the uniqueness of the solution 1 phase shifts (either $T=1$ or $T=0$). The (p,p) χ^2 surface, as outlined above, has a deep valley corresponding to solution 1, and all other solutions lie at considerably higher values.

The question still remains as to the appearance of the χ^2 surface for the combined (p,p) plus (n,p) data. In particular, if the (p,p) data are sufficient to specify the $T=1$ phase shifts, will the (n,p) data then be sufficient to specify the $T=0$ phase shifts? To investigate this question, we combined the *Harvard* and *137 MeV* (n,p) data sets. The $T=1$ phase shifts were assumed to be the same for both (p,p) and (n,p) data. The energy was taken as 142 MeV (but no data energy-shifting was performed). First, 15 solutions were chosen that fit the (n,p) data but not the (p,p) data. These were taken from the 12-phase *137-MeV* (n,p) analysis described above. When used as starting points in the combined search problem, four of them went into solution type 1 with $309<\chi^2<462$. One went into a different solution with $\chi^2=671$. Two went into solution 2 with $\chi^2=1330$ and 1440, and the rest ended with χ^2 values of 1200 or more. Problems started right on published solution 1 and solution 2 values[10] went to $\chi^2=323$ and 1422, respectively. The four solution-type 1 phase-shift sets all had approximately the same $T=0$ and $T=1$ phase shifts. Hence the combined (p,p) plus (n,p) data *are* sufficient to give a single, well-defined, solution that is the best fit to the data. As a further check, nine problems were started using the (p,p) solution types 1, 2, and 3, $T=1$ phase shifts but with completely random $T=0$ phase shifts, and using 16 phase shifts—S through F waves— in the search. Two problems went to solution 1 with $\chi^2=222$, two went to solution 2 with $\chi^2=331$ and 364, and the rest had $\chi^2>500$. (Note the lower χ^2 values for 16-phase searches as against 12-phase searches.) Thus our results are in complete agreement with earlier Dubna results[10] (which were based on incomplete data), and with earlier Harwell results[9] [which used only part of the search program outlined here, since the $T=1$ phase shifts were held fixed in fitting (n,p) data]. The conclusion is that the combined (p,p) and (n,p) analysis gives a single best solution for *both* $T=0$ and $T=1$ phase shifts.

IV. DETERMINATION OF THE $T=1$ PHASE SHIFTS FROM (p,p) DATA

Set A (with two values of σ and two of P removed) was used to determine how many non-OPEC phase shifts are needed to represent the (p,p) data. The value $g^2=14.4$ was used for OPEC. This set had 122 data points. Hence for 10 free parameters one would expect a χ^2 value of around 112 (if sufficient data smoothing has been accomplished). Nine free phase shifts—S through F waves— gave $\chi^2=157$. Freeing $^1G_4$ gave $\chi^2=137$. Freeing ϵ_4 but not $^1G_4$ gave $\chi^2=118$. Freeing both ϵ_4 and $^1G_4$ gave $\chi^2=117$. Freeing the higher phases had little effect. It is evident that at 140 MeV ϵ_4 is not adequately represented by OPEC, $^1G_4$ is only approximately represented by OPEC, and the higher phase shifts are consistent with OPEC. Hence for an accurate (p,p) phase shift analysis, at least ten phase shifts including ϵ_4 should be treated phenomenologically.

The question of how different data selections affect the phase shifts was studied by comparing solutions obtained using data sets Harvard, Harwell, *Set A*, *Set B*, and *Set C*, as listed in Table II. The phase-shift solutions are listed in Table III. The notable fact about these solutions is that although the data selections are different and the χ^2 values are radically different, the phase-shift sets are very similar. This means that the physical content of each of these data selections is approximately the same, and an attempt to choose among them purely on a χ^2 basis would be misleading. Also, the Harvard-Harwell discrepancy over cross-section shape does not strongly affect the values of the phase shifts. For an accurate determination of two-pion effects in the scattering, however, the phase-shift differences exhibited in Table III are important.

The effect of assigning an energy dependence to the phase shifts is illustrated in Table IV. The first column gives a solution for *Set A* in which the data are put in at energies ranging from 137.5 to 155 MeV, but the phase shifts are assigned no energy dependence—they have the form $\delta=\delta(a)$—and hence represent some kind of average over this energy region. The next three columns show the same problem run with a linear energy-dependent form—$\delta=\delta(a+k^2b)$—for the phase shifts (k is the c.m. momentum of a nucleon). As can be seen, the constant b is very small in all cases, and the value of χ^2 is not changed appreciably by assigning this energy dependence. Hence, while energy dependence is

TABLE IV. (p,p) phase shifts for energy-dependent and energy-independent parametrizations, and for changes in g^2 and in the number of free phase shifts.

Data selection	*Set A*	*Set A*			*Set A*	*Set A*	*142 MeV* (p,p)
Phase shift form	$\delta(a)$	$\delta(a+k^2b)$			$\delta(a+k^2b)$	$\delta(a+k^2b)$	$\delta(a)$
No. of data points	126	126			126	126	94
χ^2	165	164			160	158	114[b]
g^2	14.4	14.4			13.0	13.0	13.0
Data energy spread (MeV)	137.5–155	137.5–155			137.5–155	137.5–155	142
Phase-shift energy (MeV)	···	137.5	142	155	142	142	142
$^1S_0$	15.36	15.57	15.58	15.60	16.38	16.53	16.76
$^1D_2$	5.02	4.96	4.96	4.96	5.04	4.86	4.98
$^1G_4$	(0.54–0.60)[a]	(0.54)[a]	(0.56)[a]	(0.60)[a]	(0.50)[a]	0.67	(0.50)[a]
$^3P_0$	5.76	5.93	5.93	5.94	6.54	6.31	6.73
$^3P_1$	−17.63	−17.62	−17.62	−17.62	−17.73	−17.74	−17.74
$^3P_2$	14.07	14.07	14.07	14.07	14.22	14.28	14.18
ϵ_2	−2.61	−2.60	−2.60	−2.60	−2.63	−2.61	−2.65
$^3F_2$	0.49	0.44	0.44	0.44	0.20	0.30	0.21
$^3F_3$	−1.86	−1.83	−1.83	−1.83	−1.76	−1.77	−1.76
$^3F_4$	0.96	0.95	0.95	0.95	0.81	0.86	0.81
ϵ_4	−0.61	−0.55	−0.56	−0.59	−0.52	−0.60	−0.51
$^3H_4$	(0.20–0.24)[a]	(0.20)[a]	(0.21)[a]	(0.24)[a]	(0.19)[a]	0.20	(0.19)[a]

[a] Calculated from OPEC.

[b] Three data points contributed 27 to this χ^2 sum.

TABLE V. (n,p) phase shifts obtained after a series of variations of g^2 (see text, Sec. V).

Data selection	*Set E*	*Set E*	*Set E*	*Set E*	*Set D′*	*137 MeV* (n,p)
No. of data points	103	103	103	103	230	88
χ^2	127	123	157	119	286	138
g^2	13.0	13.0	13.0	13.0	13.0	14.4
Starting point	INITIAL	FINAL	INITIAL	FINAL	SOL. 1	RANDOM
$^1S_0$	15.3	14.9	18.9	17.9	16.6	−38.4
$^1D_2$	1.1	0.4	5.1	−1.3	5.0	7.4
$^3P_0$	2.5	5.2	4.4	9.9	6.1	−18.5
$^3P_1$	−17.3	−18.0	−17.5	−15.0	−17.8	−4.2
$^3P_2$	10.5	9.0	13.8	7.8	14.2	3.2
ϵ_2	0.1	0.2	−2.7	−0.9	−2.6	−5.3
$^3F_2$	(1.3)[a]	(1.3)[a]	0.5	−1.2	0.4	−0.8
$^3F_3$	(−2.3)[a]	(−2.3)[a]	−1.6	−2.5	−1.8	−1.3
$^3F_4$	(0.3)[a]	(0.3)[a]	0.7	0.0	0.9	1.8
ϵ_4	(−0.8)[a]	(−0.8)[a]	−1.2	−1.6	−0.6	(−0.9)[a]
$^1P_1$	−9.4	−6.4	−15.6	−4.8	−16.4	23.4
$^1F_3$	(−2.8)[a]	(−2.8)[a]	−1.9	−2.5	−1.7	2.0
$^3S_1$	26.6	26.3	28.7	28.5	28.9	32.2
ϵ_1	1.8	2.4	2.2	3.1	1.3	16.2
$^3D_1$	−12.5	−11.9	−14.1	−11.2	−14.4	−11.5
$^3D_2$	30.1	30.7	23.7	31.2	23.5	5.2
$^3D_3$	1.1	1.4	2.6	2.9	2.2	3.2..

[a] OPEC phase shifts.

TABLE VI. Error matrix calculations, using data sets *142 MeV* (p,p) and *142 MeV* $(p,p)+(n,p)$ with the error-matrix method (EMM), and *Set A′* with the parabolic-error method (PEM).

Data set	*142 MeV* (p,p)	*142 MeV* (p,p)	*142 MeV* (p,p)	*142 MeV* $(p,p)+(n,p)$	*Set A′*	*Set A′*
Method	EMM	EMM	EMM	EMM	PEM	PEM
χ^2	111.5	111.1	108.2	203.5		
g^2	(13)[a]	12.94±2.26	12.66±3.13	12.95±2.63	(11)	(13)
$^1S_0$	16.02±0.59	16.01±0.60	16.11±0.60	15.81±0.63	···	···
$^1D_2$	4.94±0.17	4.94±0.23	4.80±0.21	4.99±0.32	···	···
$^1G_4$	(0.50)	(0.50)	0.67±0.14	(0.50)	0.63±0.12	···
$^3P_0$	6.56±0.56	6.56±0.59	6.44±0.58	6.21±0.48	···	···
$^3P_1$	−17.49±0.18	−17.49±0.20	−17.52±0.17	−17.57±0.14	···	···
$^3P_2$	14.08±0.18	14.08±0.12	14.08±0.10	14.05±0.10	···	···
ϵ_2	−2.53±0.09	−2.53±0.11	−2.56±0.06	−2.55±0.10	···	···
$^3F_2$	0.21±0.27	0.21±0.27	0.28±0.29	0.34±0.21	0.24±0.32	0.15±0.29
$^3F_3$	−1.70±0.19	−1.70±0.19	−1.74±0.17	−1.76±0.17	···	···
$^3F_4$	0.84±0.15	0.84±0.15	0.94±0.20	0.89±0.11	···	···
ϵ_4	−0.48±0.05	−0.48±0.08	−0.51±0.07	−0.51±0.11	−0.54±0.07	−0.55±0.06
$^3H_4$	(0.19)	(0.19)	0.24±0.11	(0.18)	(0.16)	(0.19)
$^1P_1$				−16.33±1.49		
$^1F_3$				−1.99±0.36		
$^3S_1$				29.91±0.70		
ϵ_1				2.00±0.70		
$^3D_1$				−14.27±0.58		
$^3D_2$				22.47±0.97		
$^3D_3$				2.62±0.74		
ϵ_3				4.91±0.36		

[a] Quantities in brackets were held fixed.

of importance in treating the experimental data correctly, it is not necessary to make the phase shifts energy-dependent over this narrow band of energies. The values for b obtained here do not agree well with the results of the energy-dependent analyses.[3–5] For the final phase-shift values, an energy-dependent form was *not* used.

Column five of Table IV gives the phase shift solution for $g^2=13.0$. Column six gives the result for 12 free phase shifts instead of 10. Column seven is for the *142 MeV* (p,p) set.

Final values for the $T=1$ phase shifts are listed in Table IX and are discussed in Sec. IX.

V. ANALYSIS OF THE (n,p) DATA

The (n,p) data were analyzed by using *Set E* and searching down on phase shifts in the following manner. The coupling constant g^2 was given a series of values in succession—13, 15, 13, 11, 13, 15, 13, 11, 13, 17, 13, 9, 13— and for each of these values the phase shifts were searched, with the final phases from one run providing the starting phases for the next. When this kind of a search procedure was used for the (p,p) problems, the various solutions for $g^2=13$ were all in agreement. However, when just the (n,p) data were used, if 17 phase shifts were searched on (S–F waves), the various solutions would systematically wander away from the solution type 1 starting point. If only 12 phase shifts were searched on (S–D waves), then the solution would not wander. These results are illustrated in Table V. Columns 1 and 2 illustrate the stability of the 12-phase solution. Columns 3 and 4 show how the 17-phase solution wanders away from its starting point. The solutions labeled *initial* were started from a type 1 solution. The solutions labeled *final* were started from the penultimate solution in the g^2 sequence of search problems described above. The *Set D′* column is for combined (p,p) and (n,p) data (to be described in the next section) and gives accurate values for the $T=1$ and T=0 phase shifts for comparison purposes. From Table V we conclude that the (n,p) data give qualitative information about solution 1 but are not sufficient to give good quantitative values for the phase shifts. The S waves and the large triplet phase shifts are given with some accuracy, but little information is obtained about the small phase shifts. And of course, as was described in Sec. III, if random starting points are used, the (n,p) data will give solutions that have reasonable χ^2 values but that bear little resemblance to the solutions listed in Table V. An example of such a solution is given in the last column of Table V.

From the results of this section, it is apparent that the most accurate values for the $T=0$ phase shifts will be obtained by analyzing the (n,p) and (p,p) data simultaneously, as described in the next section.

VI. COMBINED (n,p) AND (p,p) DATA ANALYSIS

The data for this analysis were obtained by using *Set D* and *Set D′* (Table II). Charge independence was assumed in that the same $T=1$ phase shifts were used for both the (n,p) and (p,p) data. *Set D* was used to run a sequence of search problems at various values of g^2, as was done for *Set E* (Sec. V). The phase shifts now did not wander away from the starting point (solution type 1), in contrast to the situation when *Set E* was used. The combined (p,p) and (n,p) data were sufficient to accurately determine both the $T=0$ and the $T=1$ phase shifts. Also, a problem was run in which the $T=1$ phase shifts were first held fixed at values obtained from the (p,p) analysis and eight $T=0$ phases were searched. This gave a value $\chi^2=342$. Then eleven $T=1$ phases were released and all nineteen phase shifts were searched. χ^2 dropped only to 339. This shows that the $T=1$ phase shifts from the (p,p) analysis are compatible with the (n,p) data.

For the best values of the combined shifts, *Set D′* was used. This set contained 127 (p,p) data points and 93 (n,p) data points. The sets of phase shifts representing the combined analysis are listed in Table V (Set D′) and in Table IX. A further discussion of these phase shifts is given in Sec. IX.

VII. ERROR-MATRIX CALCULATIONS

The energy-dependent phase-shift code initially did not have an error-matrix routine. Hence the monoenergetic data selections *142 MeV* (p,p) and *142 MeV* (p,p) *plus* (n,p) were used to compute error matrices, using an existing monoenergetic phase-shift code and the standard error-matrix method[37] (EMM). The results of the error-matrix calculations are given in Table VI. The first data column gives the phase-shift errors for 10 free phase shifts when g^2 is held fixed. The second column shows the same calculation, but with g^2 free. As can be seen, this additional freedom does not appreciably change either the phase-shift values or phase-shift uncertainties. The third column shows the calculation when 12 phase shifts and g^2 are free. Again the change is rather small. The fourth column gives the result for the combined (p,p) plus (n,p) analysis.

In order to determine the exact significance of the error-matrix results, we computed some phase-shift errors by a different method—the parabolic-error method (PEM). In PEM, a particular phase shift δ is fixed at a certain value, and the other phenomenological phases are searched on until a minimum value for χ^2 is obtained. Then other values for δ are assigned and the procedure repeated. The resulting $\chi^2(\delta)$ versus δ curve was found to be accurately parabolic over a range of values if the initial set of phases are at the solution minimum for χ^2. The value for δ is given by the bottom of the parabola, and the error in δ is given by the width $\Delta\delta$ obtained when χ^2 is increased by one. The relationship between PEM and the error-matrix method (EMM) can be seen in the following way. EMM determines the error in a particular phase shift by computing the second derivatives

$$\partial^2\chi^2/\partial\delta_i\partial\delta_j$$

for phase-shift variations that increase χ^2 by one. A diagonal element of the inverse of this matrix gives the "correlated error" in a particular phase shift due to the correlated uncertainties in all of the phase shifts. PEM assigns a phase shift a particular value and then measures the ability of all of the other phenomenological phase shifts to change and accommodate this particular value. If the other phase shifts have large uncertainties, they can adjust easily, and the $\chi^2(\delta)$ versus δ curve will be very flat, giving a large error for δ. Similarly, if EMM has large off-diagonal elements (error correlations) for a particular phase shift, it will increase the error in that particular phase shift.

It is perhaps not immediately apparent that EMM and PEM in an actual calculation will both give the same quantitative determinations of the error in a parameter. To establish this fact, we have listed some PEM results in the last two columns of Table VI. The results in Table VI show that PEM and EMM are in substantial agreement, that changing the value of g^2 does not alter the errors appreciably, and that changing the number of free parameters does not alter the errors significantly (except perhaps for the highest coupled phase shifts). One advantage of PEM over EMM is that the correlated error for a single parameter can be directly determined without having to compute a complete matrix.

The errors for the final phase-shift values shown in Table IX were obtained with an error-matrix calculation using the energy-dependent phase-shift computer program.

VIII. DETERMINATION OF THE PION-NUCLEON COUPLING CONSTANT g^2 AND THE PION MASS μ

The pion-nucleon coupling constant g^2 can be determined by fixing g^2 successively at a series of values and searching on phase shifts to obtain a minimum least-squares sum χ^2 for each value of g^2. This is just the parabolic error method described in Sec. VII. The "correct value" for g^2 is the minimum point in the parabola. The uncertainty in g^2 is given by increasing χ^2 by 1 over the minimum value. This is the PEM error calculation. An alternate method of finding g^2 is to include it as a free parameter in the search problem and let it be varied along with the phenomenological phase shifts. The search gives the value for g^2, and the error-matrix method EMM gives the uncertainty in g^2. The equivalence of these two procedures is discussed in Sec. VII. We have carried out both of these procedures for the (p,p), the (p,p) plus (n,p) and the (n,p) systems.

The g^2 determinations for the (p,p) and for the (p,p) plus (n,p) data selections are listed in Table VII. The g^2 values are in agreement within the error limits shown, but the values are affected by the number of free phase shifts used and by the way in which the data are handled. Our conclusion is that the nucleon-nucleon data near 142 MeV indicate a g^2 value of about 12.

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[37] H. L. Anderson, W. C. Davidon, M. Glicksman, and U. E. Kruse, Phys. Rev. **100**, 279 (1955).

TABLE VII. Determinations of the pion-nucleon coupling constant g^2 from (p,p) and from $(p,p)+(n,p)$ data.

Data set	Phase-shift form	No. of free phase shifts	Type of g^2 determination	Pion mass (MeV)	g^2 value determined
142 MeV (p,p)	$\delta(a)$	10	Search	135.04	12.9±2.3[a,c]
142 MeV (p,p)	$\delta(a)$	12	Search	135.04	12.7±3.1[a,c]
142 MeV $(n,p)+(p,p)$	$\delta(a)$	18	Search	136.5	12.9±2.6[a,c]
Set A′	$\delta(a+k^2b)$	11	Parabola	135.04	11.1±2.4[b,c]
Set A′	$\delta(a)$	11	Parabola	135.04	11.8±2.1[b,d]
Set A′	$\delta(a)$	12	Parabola	135.04	8.9±3.6[b,c]
Set A′	$\delta(a)$	14	Parabola	135.04	9.2±6.9[b,c]
Set D′	$\delta(a)$	19	Parabola	136.5	12.4±2.0[b,d]

[a] Errors are from EMM (Sec. VII).
[b] Errors are from PEM (Sec. VII).
[c] Grid-search method.
[d] Signell search method (Ref. 39).

TABLE VIII. Determination of the pion mass μ from $\chi^2(\mu)$ versus μ values at different values for g^2.

Data selection	No. of (p,p) Data	No. of (n,p) Data	μ(MeV)	g^2					
				9	11	13	14.4	15	17
Set A	126	···	100	164	164	170		181	198
			135	163	160	160		161	164
			170	168	165	162		160	159
Set D′	127	103	100	286	289	311		366	458
			136.5	299	284	281		295	327
			170	325	305	289		283	284
Harvard	91	···	100				71		
			135				63		
			170				68		

However the same kind of analysis performed on nucleon-nucleon data at 210 MeV (to be published) indicates a g^2 value closer to 14. Hence it is not clear just which value for g^2 one should adopt. In Table IX we list solutions for several values of g^2. In Sec. X on charge independence, further discussion of Table VI is given.

The above analysis was also attempted using just (n,p) data. $\chi^2(g^2)$ curves were obtained, using *Set E*, that showed a minimum near $g^2=13$, but due to the tendency of the solution to wander away from a type 1 solution, these curves could not be accurately determined. When the *142 MeV* (n,p) set was used in a search problem, searching on g^2 and eighteen phase shifts, a value of g^2 close to zero was obtained. One should only conclude from this that the (n,p) data by themselves are not yet sufficient to permit this kind of data analysis.

A procedure similar to the $\chi^2(g^2)$ minimization can be used to determine the pion mass μ.[38] Since both μ and g^2 enter into OPEC, $\chi^2(\mu)$ should be investigated for several values of g^2. The results of such a calculation are given in Table VIII. As can be seen from this table, when g^2 has the value 13, the $\chi^2(\mu)$ curve has a minimum at about the expected value of 135 MeV. However, this minimum is very shallow and is useful only as a consistency check on the OPEC approximation.

IX. FINAL PHASE-SHIFT VALUES

In this section we give our best values for the phase shifts and compare these values with the results of other analyses. The best data selections are *Set A′* for the (p,p) analysis and *Set D′* for the combined (p,p) plus (n,p) analysis. As shown in Table VII, we were unable to obtain a precise value for the pion-nucleon coupling constant g^2. Our results here favor a value of about $g^2=12$, whereas pion-nucleon scattering experiments favor a value of about 15, and other phase-shift analyses at 210 MeV (to be published) favor a value of perhaps 14. Accordingly, in Table IX we list the phase-shift values for $g^2=11$, 13, and 15 for the *Set A′* and *Set D′* data selections. In Table X we list the $g^2=13$ solutions again, and compare them with the results of other investigations.[5,9,39,40,41] The Harwell[9] and early Dubna[10] analyses were discussed in the introduction (Sec. I). The more recent Dubna results[41] are influenced by a different (p,p) differential cross section[11] and by the use of OPEC to calculate ϵ_4, and hence should not be expected to agree exactly with the present analysis.

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[38] P. Signell, Phys. Rev. Letters **5**, 474 (1960).

[39] P. Signell and D. L. Marker, Phys. Rev. **134**, B365 (1964). We would like to thank Professor Signell for sending us this and other preprints on the nucleon-nucleon problem by this group, and for sending us data and other information in advance of publication.

[40] See Refs. 3 and 4. Tabulated values were obtained from the American Documentation Institute.

[41] Yu. M. Kazarinov, V. S. Kiselev, and I. N. Silin, Zh. Eksperim. i Teor. Fiz. **45**, 637 (1963) [English transl.: Soviet Phys.—JETP **18**, 437 (1964)].

The fact that the agreement is as close as it is between the $T=0$ phase shifts from Ref. 10 and from Ref. 41 indicates that most of the physical content of the nucleon-nucleon scattering experiments is contained in the subset of data used in Ref. 10, and the additional experiments used in Ref. 41 serve mainly to help reject extraneous solutions.

The Penn State phase shifts[39] were communicated to us by Professor Signell just at the conclusion of the present work. This analysis is very similar to the present analysis for the (p,p) system. In the Penn State analysis, the Harvard differential cross section[11] was used, and it was normalized to measured total cross sections.[13] One of the Penn State solutions used both the Harvard[11] and Harwell[12] $P(\theta)$ measurements. For most of the Penn State solutions, the Harwell $P(\theta)$ was excluded.

TABLE IX. Final phase-shift values from the present analysis.

Data selection	*Set A′* (See Table XI)			*Set D′* (See Table XI)		
No. of Data	127	127	127	220	220	220
$M\pi$ (MeV)	135.04	135.04	135.04	136.5	136.5	136.5
g^2	11	13	15	11	13	15
χ^2	113.6	113.8	115.4	235.8	235.4	237.0
$^1S_0$	16.71±0.57	16.65±0.58	16.57±0.59	16.67±0.54	16.62±0.55	16.57±0.55
$^1D_2$	4.98±0.20	4.92±0.20	4.85±0.20	5.05±0.19	4.97±0.19	4.89±0.19
$^1G_4$	0.62±0.12	0.64±0.12	0.66±0.12	0.59±0.11	0.60±0.11	0.62±0.11
$^3P_0$	6.40±0.58	6.34±0.59	6.27±0.60	6.07±0.51	6.09±0.51	6.09±0.51
$^3P_1$	−17.70±0.17	−17.79±0.18	−17.86±0.18	−17.76±0.14	−17.81±0.14	−17.86±0.14
$^3P_2$	14.26±0.12	14.27±0.12	14.27±0.12	14.21±0.10	14.22±0.10	14.22±0.10
ϵ_2	−2.56±0.09	−2.60±0.09	−2.63±0.09	−2.60±0.08	−2.63±0.08	−2.66±0.08
$^3F_2$	0.21±0.29	0.27±0.28	0.33±0.28	0.35±0.23	0.38±0.23	0.42±0.22
$^3F_3$	−1.74±0.19	−1.79±0.19	−1.82±0.19	−1.81±0.17	−1.84±0.16	−1.87±0.16
$^3F_4$	0.82±0.16	0.86±0.16	0.89±0.16	0.85±0.13	0.87±0.12	0.89±0.12
ϵ_4	−0.54±0.06	−0.58±0.06	−0.62±0.06	−0.57±0.06	−0.60±0.06	−0.63±0.06
$^1P_1$				−17.34±1.29	−16.40±1.30	−15.36±1.30
$^1F_3$				−1.80±0.51	−1.67±0.53	−1.57±0.55
$^3S_1$				28.89±0.65	28.92±0.66	28.92±0.67
ϵ_1				1.41±0.71	1.31±0.69	1.23±0.67
$^3D_1$				−14.33±0.52	−14.40±0.53	−14.45±0.53
$^3D_2$				23.28±0.89	23.54±0.88	23.79±0.85
$^3D_3$				2.57±0.43	2.20±0.42	1.84±0.42
ϵ_3				4.17±0.37	4.07±0.38	3.98±0.38

TABLE X. Comparison of different phase-shift determinations.

Analysis Laboratory & data selection	Present work		Energy-independent			Energy-dependent	
	Livermore		Dubna	Penn State	Harwell	Yale	Livermore
	Set A′	*Set D′*	Ref. 41	Ref. 39	Ref. 9	Ref. 40	Ref. 5
No. of Data	127	220	124	103	114	(p,p)YLAM	Midpop
χ^2	114	235	107	79.2	125.4	(n,p)YLAN3M	best sol.
g^2	13	13	11.7±1.7	14.4	14	14	14.4
$^1S_0$	16.65±0.58	16.62±0.55	17.09±0.71	16.52±0.64	16.0	13.60	17.30
$^1D_2$	4.92±0.20	4.97±0.19	5.40±0.24	4.99±0.19	5.7	5.64	4.68
$^1G_4$	0.64±0.12	0.60±0.11	0.71±0.15	0.83±0.11	0.5		0.96
$^3P_0$	6.34±0.59	6.09±0.51	6.29±0.62	6.26±0.59	6.8	4.26	6.11
$^3P_1$	−17.79±0.18	−17.81±0.14	−18.23±0.23	−17.35±0.23	−17.1	−17.10	−17.00
$^3P_2$	14.27±0.12	14.22±0.10	14.51±0.16	13.88±0.15	14.1	14.09	13.69
ϵ_2	−2.60±0.09	−2.63±0.08	−2.65±0.15	−2.63±0.13	−2.7	−2.69	−2.39
$^3F_2$	0.27±0.28	0.38±0.23	−0.00±0.32	0.51±0.36	−0.3	0.55	0.12
$^3F_3$	−1.79±0.19	−1.84±0.16	−1.72±0.23	−1.78±0.21	−1.1	−2.52	−1.42
$^3F_4$	0.86±0.16	0.87±0.12	0.45±0.20	1.05±0.20	0.2	0.40	0.65
ϵ_4	−0.58±0.06	−0.60±0.06		−0.57±0.06	−0.5		−0.79
$^3H_4$					0.2		0.06
$^1P_1$		−16.40±1.30	−12.80±3.02		−15.6	−16.53	
$^1F_3$		−1.67±0.53	−1.43±0.06		−3.6	−3.12	
$^3S_1$		28.92±0.66	28.48±0.84		29.4	29.23	
ϵ_1		1.31±0.69	−2.18±1.17		0.2	4.22	
$^3D_1$		−14.40±0.53	−15.21±0.79		−15.2	−14.87	
$^3D_2$		23.54±0.88	23.60±1.30		24.5	22.35	
$^3D_3$		2.20±0.42	−1.15±0.95		1.1	2.04	
ϵ_3		4.07±0.38	2.29±0.80				
$^3G_3$			−3.84±0.66				
$^3G_4$			4.30±0.13				
$^3G_5$			−0.51±0.36				

TABLE XI. Data used in final (p,p) (*Set A'*) and combined (p,p) plus (n,p) (*Set D'*) phase-shift analyses.

(a) (p,p) Data

$\sigma(\theta)$ at 155 MeV[a]

C. M. angle (deg.)	Datum	Exper. error
10.07	3.95	0.09
12.08	3.37	0.09
14.08	3.30	0.13
16.13	3.35	0.13
18.12	3.49	0.14
20.13	3.66	0.14
26.00	3.62	0.06
27.00	3.84	0.06
29.00	3.75	0.06
31.02	3.87	0.05
35.08	3.85	0.05
37.05	3.74	0.09
41.08	3.88	0.05
46.10	3.83	0.05
51.12	3.82	0.05
62.00	3.70	0.04
72.00	3.71	0.04
82.03	3.67	0.04
90.03	3.71	0.06
102.03	3.75	0.06
112.00	3.76	0.06

$P(\theta)$ at 147 MeV[b]

C. M. angle (deg.)	Datum	Exper. error
6.20	−0.004	0.014
8.34	0.045	0.014
10.40	0.103	0.014
12.40	0.126	0.011
14.50	0.155	0.014
16.60	0.180	0.010
18.70	0.193	0.015
20.70	0.198	0.009
22.80	0.183	0.015
24.90	0.227	0.014
31.10	0.228	0.009
25.90	0.203	0.011
36.30	0.247	0.011
41.40	0.239	0.006
46.50	0.233	0.006

$P(\theta)$ at 147 MeV[b]

C. M. angle (deg.)	Datum	Exper. error
51.70	0.229	0.006
56.80	0.205	0.006
61.90	0.171	0.006
67.00	0.154	0.006
72.00	0.131	0.006
77.10	0.098	0.006
82.10	0.052	0.008
87.20	0.030	0.008
92.20	−0.006	0.009
97.10	−0.041	0.007
102.10	−0.068	0.008
107.10	−0.109	0.008
112.00	−0.144	0.009

$D(\theta)$ at 142 MeV[c]

C. M. angle (deg.)	Datum	Exper. error
12.40	−0.262	0.063
20.70	0.008	0.038
31.10	0.137	0.033
41.40	0.156	0.031
51.70	0.178	0.033
61.90	0.076	0.031
72.00	0.147	0.070
82.10	0.286	0.099

$R(\theta)$ at 140 MeV[d]

C. M. angle (deg.)	Datum	Exper. error
31.10	−0.252	0.030
41.40	−0.227	0.028
51.70	−0.271	0.035
61.90	−0.146	0.037
72.00	0.147	0.070
82.10	0.286	0.099

$R'(\theta)$ at 137.5 MeV[e]

C. M. angle (deg.)	Datum	Exper. error
43.00	0.562	0.044
52.50	0.472	0.048
62.00	0.376	0.065
72.50	0.238	0.083
82.10	0.251	0.121

$A(\theta)$ at 139 MeV[f]

C. M. angle (deg.)	Datum	Exper. error
31.10	−0.368	0.0284
41.40	−0.344	0.0277
51.70	−0.311	0.0328
61.90	−0.231	0.0451
72.00	−0.189	0.0556
82.10	−0.099	0.0789

$P(\theta)$ at 142 MeV[g]

C. M. angle (deg.)	Datum	Exper. error
8.30	0.031	0.024
9.34	0.089	0.023
10.38	0.122	0.019
12.46	0.130	0.033
14.53	0.180	0.031
16.61	0.155	0.028
31.06	0.238	0.028
20.76	0.190	0.009
25.95	0.225	0.011
31.06	0.241	0.010
37.20	0.283	0.030
41.34	0.237	0.011
82.06	0.066	0.011
90.00	0.010	0.011
24.80	0.216	0.037
45.45	0.242	0.005
49.55	0.240	0.004
51.60	0.232	0.007
53.65	0.213	0.004
57.70	0.205	0.006
59.75	0.197	0.005
61.80	0.180	0.005
65.90	0.170	0.005
69.95	0.141	0.005
72.00	0.117	0.005
74.05	0.097	0.006
82.10	0.051	0.015

$D(\theta)$ at 143 MeV[h]

C. M. angle (deg.)	Datum	Exper. error
31.06	0.082	0.077
41.34	0.162	0.040
51.62	0.110	0.050
61.84	0.045	0.060
71.98	0.019	0.100
82.06	−0.037	0.133
92.00	−0.027	0.170

$R(\theta)$ at 142 MeV[i]

C. M. angle (deg.)	Datum	Exper. error
24.00	−0.224	0.051
32.70	−0.203	0.051
45.70	−0.178	0.031
54.40	−0.212	0.042
67.20	−0.213	0.040
76.10	−0.147	0.063
84.40	−0.142	0.136
90.00	0.110	0.131

$A(\theta)$ at 142 MeV[j]

C. M. angle (deg.)	Datum	Exper. error
32.20	−0.405	0.032
43.20	−0.377	0.037
54.60	−0.342	0.050
65.00	−0.355	0.075
74.80	−0.198	0.079
84.80	0.022	0.154

$R'(\theta)$ at 140 MeV[k]

C. M. angle (deg.)	Datum	Exper. error
31.40	0.625	0.062
41.70	0.548	0.062
52.00	0.470	0.069
61.80	0.343	0.058
82.20	0.190	0.177

(b) (n,p) Data

$\sigma(\theta)$ at 128 MeV[l]

C. M. angle (deg.)	Datum	Exper. error
78.1	2.71	0.110
88.1	2.61	0.090
98.1	2.83	0.110
108.2	3.45	0.120
118.4	4.29	0.150
128.5	5.20	0.190
138.8	6.28	0.230
149.0	7.30	0.260
159.3	9.08	0.330
169.7	11.37	0.410

$P(\theta)$ at 126 MeV[m]

C. M. angle (deg.)	Datum	Exper. error
33.0	0.436	0.066
41.2	0.446	0.039
51.7	0.571	0.041
61.7	0.588	0.044
71.8	0.471	0.044
81.9	0.312	0.051

$D_T(\theta)$ at 128 MeV[n]

C. M. angle (deg.)	Datum	Exper. error
124.0	−0.117	0.165
133.0	−0.252	0.150
142.0	−0.035	0.148
150.0	−0.013	0.117
160.0	0.174	0.146

$\sigma(\theta)$ at 130 MeV[o]

C. M. angle (deg.)	Datum	Exper. error
25.0	6.30	0.75
35.0	5.38	0.59
45.0	3.72	0.44
55.0	3.18	0.38
65.0	2.36	0.31
75.0	2.85	0.33
85.0	2.54	0.31
95.0	2.44	0.30
105.0	3.01	0.34
115.0	3.66	0.39
125.0	5.11	0.48

$\sigma(\theta)$ at 130 MeV[o]

C. M. angle (deg.)	Datum	Exper. error
135.0	5.03	0.51
145.0	5.90	0.62
155.0	8.41	0.87

$A(\theta)$ at 135 MeV[p]

C. M. angle (deg.)	Datum	Exper. error
42.1	−0.020	0.089
52.5	0.070	0.074
62.9	0.210	0.088
73.4	0.125	0.105
83.6	0.532	0.220

$\sigma(\theta)$ at 137 MeV[q]

C. M. angle (deg.)	Datum	Exper. error
6.3	9.06	1.00
10.6	8.16	0.60
20.7	7.02	0.40
31.0	5.96	0.35
41.3	4.46	0.40
51.6	2.91	0.40
61.8	2.75	0.40

$R(\theta)$ at 137 MeV[p]

C. M. angle (deg.)	Datum	Exper. error
42.1	0.169	0.100
52.5	0.080	0.093
62.9	−0.023	0.073
73.4	−0.151	0.095
83.6	−0.146	0.210

$P(\theta)$ at 140 MeV[r]

C. M. angle (deg.)	Datum	Exper. error
20.7	0.283	0.027
31.0	0.363	0.018
41.3	0.491	0.022
68.0	0.451	0.025
78.0	0.303	0.027
88.0	0.232	0.017
98.0	0.083	0.019
108.0	0.032	0.013
118.6	−0.038	0.012
128.6	−0.044	0.009
138.7	−0.059	0.009
149.0	−0.074	0.012
159.3	−0.037	0.012
61.4	0.593	0.035

[a] Normalization error (NE), 4%. Renormalization (Re): A', 1.015; D', 1.014.[14]
[b] NE, 3%. Re: A', 0.988; D', 0.980.[11]
[c] Re: A', 1; D', 1.[15]
[d] Re: A', 1; D', 1.[17]
[e] NE, 5%. Re: A', 0984; D', 0.986.[21]
[f] NE, 4%. Re: A', 1.007; D', 1.009.[19]
[g] NE, 2.2%. Re: A', 0.986; D', 0.961.[12]
[h] Re: A', 1; D', 1.[16]
[i] Re: A', 1; D', 1.[18]
[j] Re: A', 1; D', 1.[20]
[k] Re: A', 1; D', 1.[22]
[l] NE, 2.2%. Re: D', 1.002.[25]
[m] NE, 4%. Re: D', 0.972.[25]
[n] Re: A', 1; D', 1.[32]
[o] NE, 3.2%. Re: D', 1.045.[26]
[p] Re: D', 1.[33,34]
[q] NE, 5%. Re: D', 1.025.[27]
[r] NE, 4.4%. Re: D', 0.994.[30]
[s] Re: D', 1.[31,34]
[t] NE: 2.2%. Re: D', 0.975.[28]

TABLE XI (*continued*)

C. M. angle (deg.)	Datum	Exper. error	C. M. angle (deg.)	Datum	Exper. error
(b) (n,p) Data					
$P(\theta)$ at 143 MeV[s]			$\sigma(\theta)$ at 153 MeV[t]		
41.0	0.526	0.056	83.0	1.98	0.19
51.0	0.526	0.045	89.5	2.29	0.18
62.0	0.478	0.045	98.0	2.71	0.31
72.0	0.392	0.045	99.5	2.51	0.19
82.5	0.226	0.045	112.0	3.87	0.18
92.5	0.111	0.045	124.5	4.04	0.28
108.0	0.015	0.045	138.0	6.19	0.26
118.0	−0.020	0.045	149.0	6.88	0.43
$\sigma(\theta)$ at 153 MeV[t]			159.0	7.98	0.13
50.0	2.96	0.43	165.0	8.59	0.29
56.0	2.14	0.40	171.0	10.04	0.20
65.5	2.59	0.40	174.0	9.68	0.47
68.0	2.34	0.18	176.0	10.65	0.46
76.5	1.98	0.22	178.0	10.69	0.54

The Penn State solution shown in Table X is an example of the latter. The agreement between this solution and the *Set A′* result is within the uncertainties for the phase shifts. The reasons that the Signell group discarded the Harwell $P(\theta)$ data[12] were that they gave a large contribution to χ^2 (2 per data point on the average) and that they differed somewhat in shape at large angles from Harvard measurements[11] and potential model calculations.[39] However our analysis including both the Harwell $P(\theta)$ data[12] (with the points at 5.2, 6.2, and 78.05° removed) and the Harvard $P(\theta)$ data[11] gave an average χ^2 contribution of about 1 per data point for the Harvard and for the Harwell data (see Sec. II). This indicates that the Harwell and Harvard $P(\theta)$ measurements are reasonably compatible and the Harwell data (except for the three excluded points) are internally self-consistent. Thus, it is not obvious to us that the Harvard polarization data are to be preferred over the Harwell polarization data. Recent polarization measurements at Orsay[42] will be helpful in clarifying this point. In any case the difference between using the Harwell (p,p) $P(\theta)$ data and not using them is not very great.

As a comparison check between our results and those of the Penn State group, we used our computer program with the Penn State data selection.[39] The Penn State solution listed in Table X gave a χ^2 value on our computer program that was within about 1% of the value obtained by Signell. Considering the extreme sensitivity of χ^2 to any variations in the parameters, this constitutes an excellent check on the two computer programs. We then ran our solution with the Signell data and the Penn State solution with the Livermore data. The results showed clearly that the small differences in phase-shift values that exist between the Livermore and Penn State solutions in Table X are due to the slightly different data selections used, and are not due to errors in programming or to differences in the search procedures.

The Yale[3,4] and Livermore[5] energy-dependent solutions have values at 142 MeV (143 MeV for YLAN3M) that are very similar to the present results, as can be seen from the last two columns of Table X. The energy-dependent analyses covered the elastic nucleon-nucleon region (roughly 0–400 MeV). The virtue of the energy-independent analysis is that we obtain precise values for the highest l phenomenological phase shifts, where there is hope of isolating specific 2π or 3π effects in the scattering.

As stated at the beginning of this paper, some of the experimental data used in this analysis were assigned over-all normalization errors. In the *Set A′* solution given in Table X, the data renormalization was 2% or less for all data except the Harwell polarization data,[12] which were renormalized downward by 3%. In the *Set D′* analysis, the (p,p) data renormalizations were almost the same as for *Set A′*. The (n,p) renormalizations were also small. In Table XI we list the data as they were used for the *Set A′* and *Set D′* solutions shown in Table X.

In most of the phase-shift searches for the present paper, we used the grid-search method, in which the phase shifts are systematically varied one at a time. This is an excellent search procedure when starting far away from a solution in phase-shift space. However, the grid method has the disadvantage that the final phase shifts are not unique but depend slightly on the starting point, with the spread in values corresponding rather directly to the freedom permitted by the error-matrix uncertainties. To obtain precise final values, a better procedure is to use a search method in which all the phase shifts are simultaneously varied in moving toward the minimum value for χ^2. This method, which is the one used by Signell,[39] was used to obtain the final phase-shift values shown in Tables IX and X. It has an additional advantage that the error-matrix calculation is automatically given at the end of the search.

One phase shift about which there is some question is ${}^3F_2$. As can be seen in Table X, this phase shift is not accurately determined. Using the grid search method, we found that ${}^3F_2$ was quite sensitive to the choice of normalization for the (p,p) $P(\theta)$ data. In particular, when we removed all constraints on the (p,p) $P(\theta)$ data, the ${}^3F_2$ error, as determined by PEM (Sec. VII), became about 0.5°. Signell,[39] using a different search procedure, did not find this sensitivity of ${}^3F_2$ to the polarization data. There is also a small difference in $P(\theta)$ polarization normalization arrived at in the various phase-shift analyses. In the Perring analysis,[9] the Harwell (p,p) polarization data[12] had to be renormalized downward by 5–7% in order to obtain good solutions. Signell,[39] using the Harvard $P(\theta)$ data,[11]

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[42] R. A. Bryan, private communication about measurements made by A. Michalowicz.

arrived at a normalization that was also about 5% downward. Our values for these two renormalizations (Table XI) were about 3% [12] and 1%.[11] The $^3F_2$ phase shift is of special importance because it has a bearing on recent theoretical nucleon-nucleon calculations.[43,44] Orsay measurements of the (p,p) polarization[42] should help to pin down the value for $^3F_2$.

X. CHARGE INDEPENDENCE

In this analysis, we have examined some of the general features of the charge independence hypothesis. We assumed that the pion-nucleon coupling constant is essentially the same for charged and uncharged pions and that the $T=1$ phase shifts are the same for both (p,p) and (n,p) scattering data. Under these assumptions we found that both (p,p) data analyzed separately, and (p,p) plus (n,p) data analyzed together, give about the same value for the coupling constant (Table VII). Also, $T=1$ phase shifts as determined by just (p,p), and then by (p,p) plus (n,p) data, agree to well within the uncertainties in the phase shifts (Tables IX and X).

An attempt to analyze just the (n,p) data (Sec. V) showed that, while the data permit a solution that is fully compatible with the (p,p) $T=1$ phase shifts, they also permit other equally good solutions that are not. This result is not surprising in view of the incompleteness of the (n,p) scattering measurements.

A determination of the pion mass (Sec. VIII) again showed that the (p,p) and the combined (p,p) plus (n,p) data give about the same result, although the effect of varying the pion mass a few MeV is so slight that this is only a qualitative conclusion.

In working with the (p,p) system we used a pion mass in the OPEC calculation of 135.04 MeV, corresponding to the π^0. For the (n,p) system, the proper pion mass to use is 138.06 MeV, an average of the three pion masses. For the combined (p,p) plus (n,p) analysis, we used an average pion mass of 136.5 MeV. This means in a practical sense that (roughly speaking) for an analysis at 142 MeV, the (p,p) OPEC phase shifts being used are at an energy of 139 MeV, and the (n,p) OPEC phase shifts are at an energy of 145 MeV. Since it has been shown here that the energy-dependent factors are not very important (Table IV), this shift of OPEC energy is a small effect. A number of other small effects, such as magnetic moment interactions, have been neglected. In view of the incompleteness of the (n,p) data, we do not feel it is justified at this time for us to consider small deviations from charge independence, since our analysis has no way of detecting these small effects. The analysis certainly indicates that the nucleon-nucleon data near 140 MeV are consistent with the gross features of the charge independence hypothesis. Breit and collaborators[45,46] have previously studied the nucleon-nucleon problem from the standpoint of charge independence and reached the same conclusions. Our work is a confirmation of their results.

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